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On a Theorem of Nordhaus and Gaddum

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Nordhaus and Gaddum [2] have shown that if G has p points and chromatic number χ and if the complementary graph $\bar{G}$ has chromatic number $\bar{\chi}$, then

$$\chi + \bar{\chi} \leqslant p + 1, \tag{1}$$

$$\chi\bar{\chi} \geqslant p. \tag{2}$$

THEOREM. *For every pair of positive integers n and $\bar{n}$ satisfying (1) and (2) there exists a graph G with p points for which $\chi = n$ and $\bar{\chi} = \bar{n}$.*

PROOF: Under conditions (1) and (2), p may be partitioned into $\bar{n}$ positive integers of which at least one is n and none of which is larger than n. For from (1) we have $p - n \geqslant \bar{n} - 1$, so that after n is subtracted from p the remainder is large enough to be divided into $\bar{n} - 1$ positive integers. Then from (2) we have $p - n \leqslant n(\bar{n} - 1)$, so that none of these $\bar{n} - 1$ additional integers needs to be larger than n.

Using such a partition of p, let G be the union of $\bar{n}$ disjoint complete graphs, G_1, G_2, ..., $G_{\bar{n}}$, none containing more than n points and at least one, say G_1, containing n points.

Then $\chi = n$. For at least n colors are needed to color G_1. And not more than n colors are needed because no G_i contains more than n points and the subgraphs are disjoint.

Also $\bar{\chi} = \bar{n}$. For $\bar{G}$ contains at least one complete subgraph with $\bar{n}$ points, because none of the G_i (nor by definition [1] any other graph) is empty. And to color $\bar{G}$ does not require more than $\bar{n}$ colors, for all the points in any one of the subgraphs G_i may be given the same color since G_i is a complete subgraph of G, hence totally disconnected in $\bar{G}$.

A more elaborate treatment of this subject appears in [3].

References

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3. H. J. Finck, On the Chromatic Numbers of a Graph and Its Complement, *Theory of Graphs, Proc. Colloq., Tihany, Hungary, 1966* (P. Erdös and G. Katona, eds.), Budapest, 1967, pp. 99–113.